3D: Stability and Regulation

This section discusses another aspect of active filters that was neglected in the preceding section. The prior section implicitly assumed that all poles of the circuit's voltage transfer function had poles with negative real parts. When this is the case, then we know the natural response (impulse response) of the circuit is the sum of exponentially decaying functions that asymptotically go to zero as t goes to infinity. Such circuits are said to be *input-output stable*. What this section does is consider a couple of situations where this input-output stability is degraded or destroyed in an active filter. To correct this problem, we need to introduce some sort of *compensation* to alter the frequency response of the op-amp.

Let us first consider the circuit shown in Fig. 1. This is a unit-gain inverting op-amp configuration that is driving a large capacitive load. We assume that the op-amp's transfer function can be modeled by a single dominant pole so it takes the form

$$a(s) = \frac{10^7}{s+10} + \frac{K_a}{s+\omega_a}$$

The feedback resistors are $R = 10$ kohm and the load capacitor is 10 μF. We initially assume we can neglect the output impedance of the op-amp and we let its input impedance be infinite.

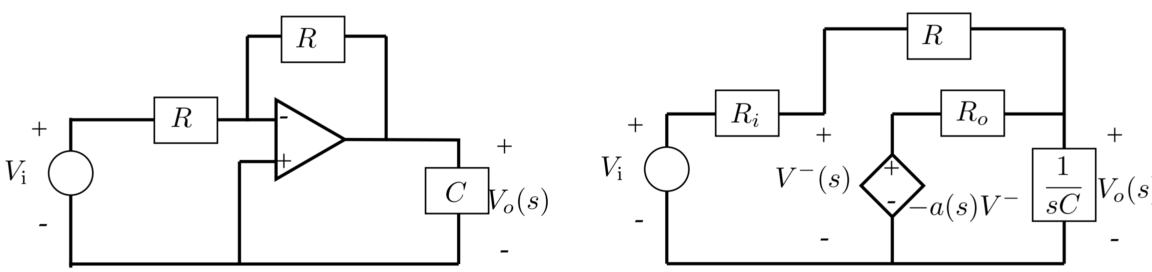

FIGURE 1. Unity Gain Inverting Amplifier driving a large capacitive load

Since we assume that the op-amp's output impedance was zero, we can use our earlier analysis to verify that

$$\begin{aligned} V_o(s) &= \frac{-1}{1+2a^{-1}(s)} V_{\rm in}(s) \\ &= \frac{-10^7/2}{s+(10^7/2+10)} V_{\rm in}(s) \end{aligned}$$

So the circuit's voltage transfer function

$$G(s) = \frac{-10^7/2}{s + (10^7/2 + 10)}$$

The pole has a negative real part and this is a low pass filter whose cutoff frequency $\omega_c = 10^7/2$ rad/sec. We can readily determine that the step response of this circuit should look like

$$v_o(t) = -\left(1 - e^{-10^7/2t}\right) u(t)$$

which would get within 5 percent of its final value in 3 time constants

$$3 \times \frac{2}{10^7} \approx 6\ \mu\text{sec}$$

So this is an extremely fast response time.

Real life op-amps, however, due not have output impedances that are zero. As an example, the commercially available OP37 op-am (analog devices) has an output impedance of 70 ohms with the same $a(s)$ that was given above. Let us see how this impacts the step response of our low pass filter.

The equivalent circuit model for our filter is shown in Fig. 1 where we added the output impedance R_o between the dependent source and the load capacitor. We assume the same parameters R, C, and $a(s)$ as before. So the only thing that has changed is the addition of the small output impedance.

Applying KCL at the V^- node gives

$$\frac{V_o - V^-}{R} = \frac{V^- - V_i}{R}$$

Since the input and feedback resistors on the op-amp are the same, this equation can be rewritten as

(1) $$V^-(s) = \frac{1}{2}V_o(s) + \frac{1}{2}V_i(s)$$

We now apply KCL at the V_o node to get

$$\frac{V^- - V_o}{R} + \frac{-a(s)V^- - V_o}{R_o} = sCV_o(s)$$

and we rearrange this to place the V^- and V_o terms on opposite sides of the equation

$$V^-(s)\left(\frac{1}{R} - \frac{a(s)}{R_o}\right) = V_o(s)\left(sC + \frac{1}{R} + \frac{1}{R_o}\right)$$

We insert equation (1) into $V^-(s)$ to get

$$\left(\frac{1}{2}V_o + \frac{1}{2}V_i\right)\left(\frac{1}{R} - \frac{a(s)}{V_o}\right) = V_o\left(sC + \frac{1}{R} + \frac{1}{R_o}\right)$$

again collecting terms to separate V_o and V_i on opposite sides gives

$$V_o\left(sC + \frac{1}{R} + \frac{1}{R_o} - \left(\frac{1}{2}\right)\left(\frac{1}{R} - \frac{a(s)}{R_o}\right)\right) = \left(\frac{1}{2}\right)\left(\frac{1}{R} - \frac{a(s)}{R_o}\right)V_i(s)$$

So the voltage transfer function becomes

$$\begin{aligned} G(s) &= \frac{V_o(s)}{V_i(s)} \\ &= \frac{\left(\frac{1}{2}\right)\left(\frac{1}{R} - \frac{a(s)}{R_o}\right)}{sC + \frac{1}{R} + \frac{1}{R_o} - \left(\frac{1}{2}\right)\left(\frac{1}{R} - \frac{a(s)}{R_o}\right)} \end{aligned}$$

We substitute $a(s) = \frac{K_a}{s+\omega_a} = \frac{10^7}{s+10}$ to get

$$G(s) = \frac{0.007s - 10^7}{0.0014s^2 + 2.021s + 10^7}$$

There is a right half plane zero, but of greater interest are the poles of the transfer function

$$p_{1,2} = -721 \pm j84512$$

How does this compare to the case when $R_o = 0$? Fig. 2 shows the step response for this filter when $C = 10$ μF$(top) and when$C = 10nF. In the top plot we see that the step response is

$$1 - e^{-712t}\cos(84512t + \phi)$$

for $t \geq 0$ which is about 7000 times *slower* than the response we computed when $R_o = 0$. In addition to this we see there is a high frequency oscillation of about 13.5 kHz. From the top plot in Fig. 2 we see that with $R_o = 70$ ohms, the response is highly oscillatory with essentially a 100 percent overshoot, whereas with $R_o = 0$ the output appears to jump instantaneousy to its final value. There is clearly a huge difference between the two responses.

The reason for this difference is that the combination of the output impedance, R_o with the load capacitor C produces a pole that is highly oscillatory. We can reduce this by using a smaller load capacitor, but the bottom plot of Fig. 2 shows that even for a small cap of

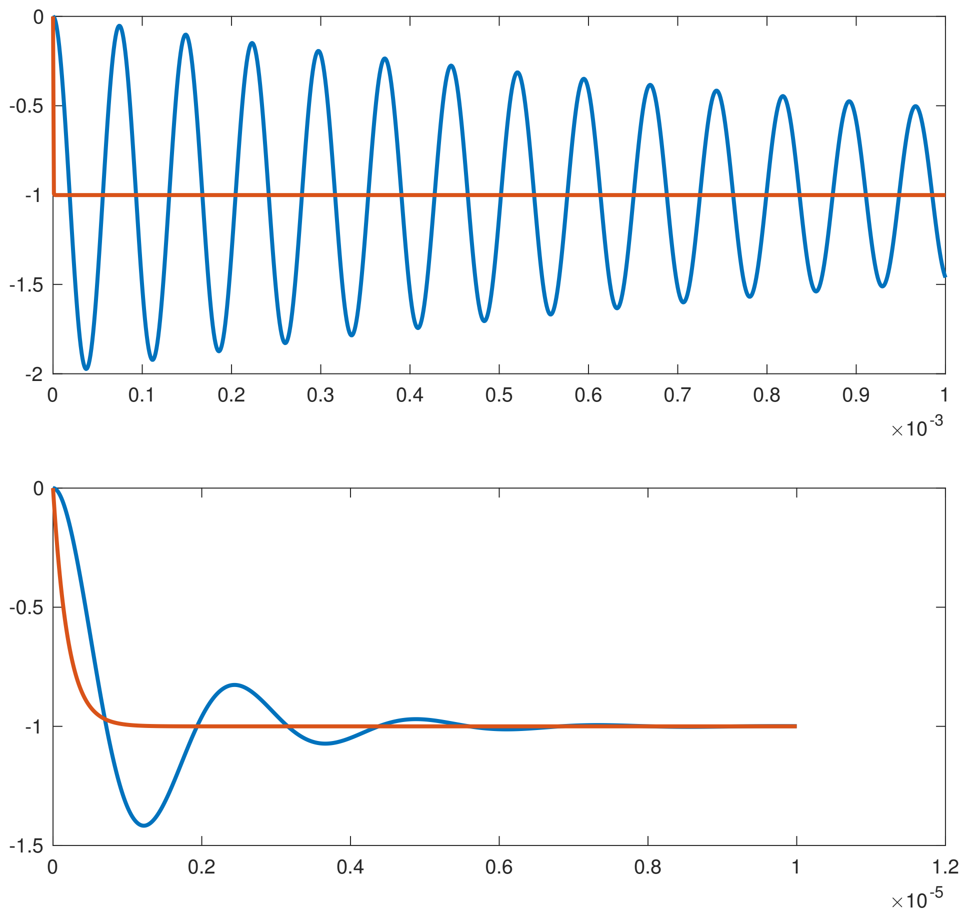

FIGURE 2. Step response of low pass active filter with $R_o = 70$ ohms for $C = 10$ μF (top) and $C = 10$ nF (bottom

10 nF the response is still has a large overshoot of 50 percent which would not usually be considered very good.

Let us take a closer look at how one might relate the poles of a transfer function to its transient response. In particular, let us assume we can write the transfer function as

$$G(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

where ξ and ω_n are two parameters. Note that our preceding transfer function's denominator polynomial can be put in this form. The parameters ξ and ω_n are called the poles *damping ratio* and *natural frequency*, respectively. These are useful parameters for they allow us to easily visualize the pole locations in the complex plane and how those pole locations are related back to the natural response of the transfer function.

The poles of the preceding transfer function are obtained through the quadratic formula,

$$\begin{aligned} p_{1,2} &= \frac{1}{2}\left(-2\xi\omega_n \pm \sqrt{4\xi^2\omega_n^2 - 4\omega_n^2}\right) \\ &= -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1} \end{aligned}$$

Note that the real part of $p_{1,2}$ is always negative so such transfer functions are always stable. But the overshoot and "oscillatory" nature of the response can vary greatly depending on the damping ratio ξ.

In particular, if $\xi > 1$, then the poles are all real. There is a *slow pole* that we may write as

$$p_{\text{slow}} = -\xi\omega_n + \omega_n\sqrt{\xi^2 - 1}$$

and a fast pole

$$p_{\text{fast}} = -xi\omega_n - \omega_n\sqrt{\xi^2 - 1}$$

The relative location of these poles is shown in Fig. 3. In general, we want a fast real pole because this means we reach the steady state quickly and for a similar reason we find the slow pole undesirable. In particular, the larger ξ is the slower the slow pole will be. When $\xi = 1$, then both poles have the same value which represents the case of *critical damping*.

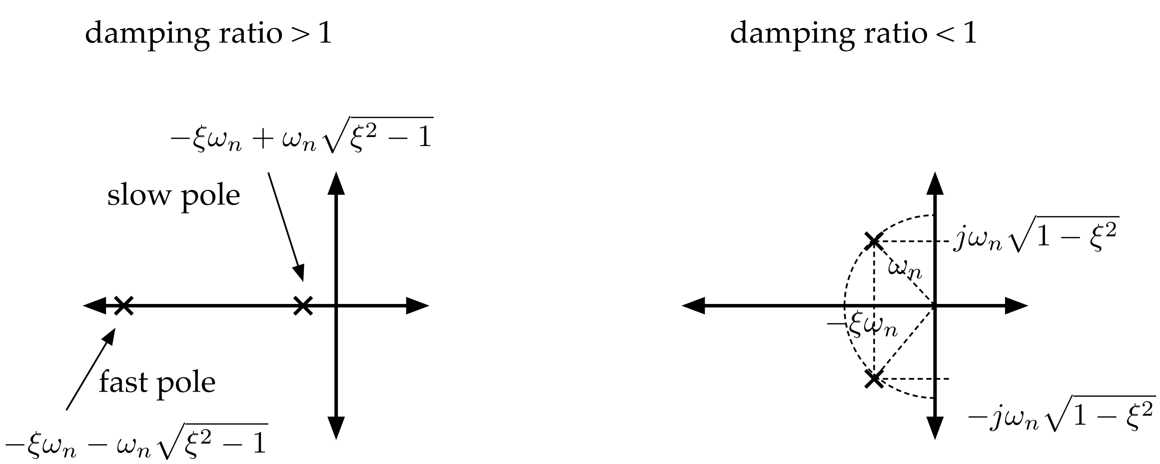

FIGURE 3. Pole Locations of second order system

When the damping ratio $xi < 1$, then the poles form a complex conjugate pair

$$p_{1,2} = -\xi\omega_n \pm j\omega_n\sqrt{1 - \xi^2}$$

These poles still have negative real parts (so the system is I/O stable), but as ξ gets smaller and smaller the poles get closer and closer to the imaginary axis. This means that we have a step response of the form,

$$v_o(t) = (1 - e^{-\xi\omega_n t}\cos(\omega_n\sqrt{1-\xi^2}t + \phi)u(t)$$

For small ξ, the exponent is very small, which implies a slow decay and the frequency of oscillation is essential ω_n. We can therefore have the case where there are many oscillations before the exponential term dies out and this is what leads to the oscillatory response. In particular, as long as $0.7 < \xi < 1$, then we can expect the overshoot of the response to be zero, with the best response (fastest rise time) occuring when $\xi = 0.7$. As ξ decreases we see the response overshoot its final value by up to 100 percent. For our example, we saw that $2\xi\omega_n \approx 2$ and $\omega_n^2 = 10^7$, which implies

$$\xi = \frac{1}{\sqrt{10^7}} = 3.16 \times 10^{-4} \ll 1$$

Since this much smaller than one, we will see nearly a 100 percent overshoot which is exactly what was shown in the responses in Fig. 2.

The response we obtained with using the "real" op-amp to drive a large capacitive load was very undesirable. In practice one would *compensate* the op-amp to reduce this oscillation. There are many different ways of achieving such compensation, but one way to do this is to introduce a capacitor in the feedback path as shown in Fig. **??**. Essentially what this capacitor does is introduce zeros that cancel the high-frequency pole introduced by the nonzero output impedance R_o.

I'm now going to turn to another problem, in which we seek to *regulate* a voltage source's output. Regulation means that we keep the output at a desired set point. This might be needed if we are using an unregulated source like a battery to drive a load that has very stringent input voltage requirements. The battery's voltage changes over time as its state-of-charge decays. The load, however, may not be able to tolerate this variation and so we introduce a op-amp circuit between the unregulated source (battery) and the load that regulates the source's voltage, $V_{\rm in}$, to generate an output voltage $V_{\rm out}$ that is close to a desired *nominal voltage*, $V_{\rm nom}$. In other words, the device (called a *voltage regulator*) ensures

$$|V_{\rm nom} - V_{\rm out}| \approx 0$$

(or at least is small). This performance of the voltage regulator is often characterized in terms of its *percent regulation*

$$\text{Percent Regulation} = 100 \times \frac{V_{\text{out,max}} - V_{\text{out,min}}}{V_{\text{nom}}}$$

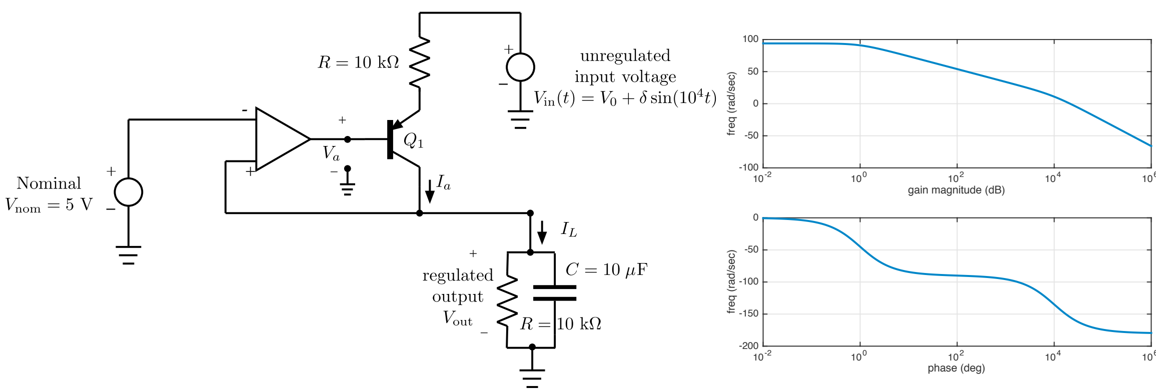

FIGURE 4. Voltage Regulator

A common circuit used for this voltage regulation job is shown in Fig. 4. To see how this works, let us apply KVL from the input voltage V_{in} to ground through the transistor and the RC load.

$$V_{\text{in}} = RI_L + a(s)(V_{\text{out}} - V_{\text{nom}})$$

Since the current going through the load is

$$I_L = \frac{V_{\text{out}}}{R\,||\,\frac{1}{sC}} = \frac{RCs+1}{R} V_{\text{out}}$$

we can readily see that increasing I_L will increase V_{out} and decreasing I_L will decrease V_{out}. In particular, our KVL equation can be rewritten as

$$RI_L = V_{\text{in}} - a(s)(V_{\text{out}} - V_{\text{nom}})$$

which suggest that if

$$V_{\text{out}} - V_{\text{nom}} > 0, \text{ then } I_L \text{ will decrease and so } V_{\text{out}} \text{ will decrease}$$

In a similar way if

$$V_{\text{out}} - V_{\text{nom}} < 0, \text{ then } I_L \text{ will increase and so } V_{\text{out}} \text{ will increase}$$

The rest point of this occurs when $V_{\text{out}} - V_{\text{nom}} = 0$, which suggests that we are using the difference between V_{out} and V_{nom} to correct the output voltage. In other words, we have a feedback system that uses our voltage error ($V_{\text{out}} - V_{\text{nom}}$) to correct the output voltage and thereby keep it at its nominal value.

With this in mind, let us see what the transfer functions are from V_{in} and V_{nom} to the voltage error $V_{\text{out}} - V_{\text{nom}}$. From the KVL equation we see that

$$V_{\text{in}} = (RCs + 1)V_{\text{out}}(s) + a(s)(V_{\text{out}} - V_{\text{nom}})$$

We rewrite the KVL equation as

$$\frac{V_{\text{in}}}{RC + 1} = V_{\text{out}} + \frac{a(s)}{RCs + 1}(V_{\text{out}} - V_{\text{nom}})$$

and for convenience let $G(s) = \frac{1}{RCs+1} = \frac{1}{10^{-1}s+1}$ so that

$$V_{\text{out}}(1 + a(s)G(s)) = V_{\text{in}}G(s) + a(s)G(s)V_{\text{nom}}$$

Dividing both sides by $1 + a(s)G(s)$ gives,

$$V_{\text{out}}(s) = \frac{G(s)}{1 + a(s)G(s)}V_{\text{in}}(s) + \frac{a(s)}{1 + a(s)G(s)}V_{\text{nom}}(s)$$

This explicitly shows us what the transfer functions are for V_{in} to V_{out} and V_{nom} to V_{out}. We want the transfer function from V_{in} to $V_{\text{out}} - V_{\text{in}}$. Simply subtracting V_{nom} from above gives

$$\begin{aligned} V_{\text{out}} - V_{\text{nom}} &= \frac{G(s)}{1 + a(s)G(s)}V_{\text{in}}(s) + \frac{a(s)G(s)}{1 + a(s)G(s)}V_{\text{nom}} - V_{\text{nom}} \\ &= \frac{G(s)}{1 + a(s)G(s)}V_{\text{in}}(s) - \frac{1}{1 + a(s)G(s)}V_{\text{nom}}(s) \end{aligned}$$

and so we see that the transfer function from the unregulated input V_{in} to the voltage error $V_{\text{out}} - V_{\text{nom}}$ is $\frac{G}{1+aG}$.

We now assume that

$$a(s) = \frac{5 \times 10^4}{(s + 1)(10^{-4}s + 1)}$$

This has a dominant pole at 1 rad/sec and a high frequency pole at 10^4 rad/sec associated with a nonzero output impedance. For the values we chose, we see

$$\begin{aligned}\frac{G(s)}{1+a(s)G(s)} &= \frac{\frac{1}{0.1s+1}}{1+\frac{5\times 10^4}{(s+1)(10^{-4}s+1)}\frac{1}{0.1s+1}} \\ &= \frac{(s+1)(10^{-4}s+1)}{(s+1)(10^{-4}s+1)(10^{-1}s+1)+5\times 10^4} \\ &= 10\frac{s^2+10001s+10000}{s^3+10010s^2+110010s+5.0001\times 10^9}\end{aligned}$$

This has zeros at -1000 and -1 rad/sec. It has a pole at -1.005×10^4 rad/sec and a complex conjugate pair of poles at

$$(0.0019 \pm j0.07075)\times 10^4$$

Note that the real part of this pole pair is positive. This means that the natural response of the output grows in an exponential manner and so this circuit is *unstable*.

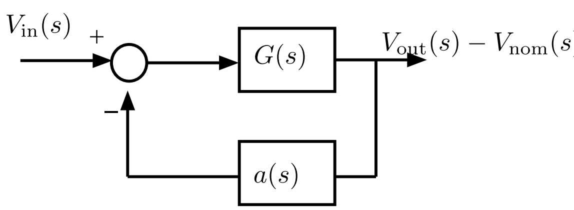

FIGURE 5. Feedback Loop for VR

Let us take a closer look to see what is going on. I'm going to rewrite this transfer function $\frac{G}{1+aG}$ as a signal flow diagram shown in Fig. 5. One can readily verify that the transfer function for this block diagram is identical to that for our Voltage Regulator. In particular, this diagram shows,

$$V_{\text{out}} - V_{\text{nom}} = G(s)\left(V_{\text{in}} - a(s)(V_{\text{out}} - V_{\text{nom}})\right)$$

We rearrange to get

$$(1+a(s)G(s))(V_{\text{out}} - V_{\text{nom}}) = G(s)V_{\text{in}}$$

which clearly shows that

$$\frac{V_{\text{out}}(s) - V_{\text{in}}(s)}{V_{\text{in}}(s)} = \frac{G(s)}{1+a(s)G(s)}$$

Now consider a harmonic in V_{in} of frequency ω. Note that at the summing functions, it is combined with the ω harmonic of $V_{\text{out}} - V_{\text{in}}$. Note that if this harmonic of $V_{\text{out}} - V_{\text{in}}$ is shifted in phase by 180 degrees, then it *re-inforces* the harnonic of V_{in}. This would then be amplified by $a(s)G(s)$ if $|a(s)G(j\omega)| > 1$, which would cascade in an increasing way to cause instability. The preceding discussion suggests that for stable operation we require all harmonics of the input that are amplified by the loop, but be re-injected with a phase less than $180°$. We can check to see if this happens for our our *loop function*

$$a(s)G(s) = \frac{5 \times 10^4}{(s+1)(10^{-4}s+1)(10^{-1}s+1)}$$

by looking at its Bode plot in Fig. 6. This plot shows that there is a range of frequencies around 700 rad/sec, where the gain is positive (greater than 0 dB) and yet the phase is less than $-180°$. This is enough to suggest instablity which is indeed what happens in this circuit.

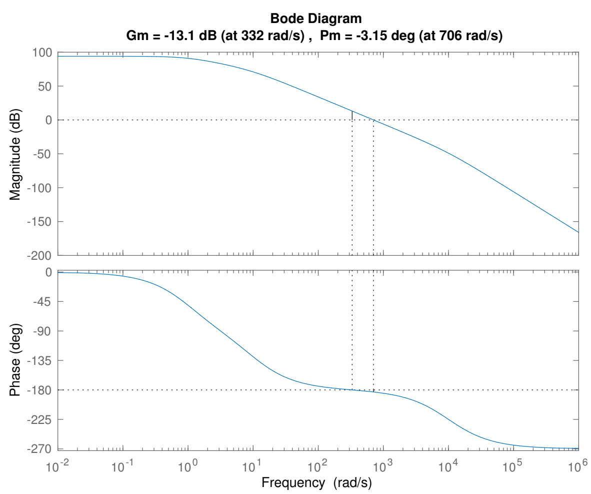

FIGURE 6. Bode Plot of aG

The plot, however, also immediately suggests a way to compensate for this. Namely, if one were to lower the gain $a(s)G(s)$ by a factor of 10, then all frequencies that amplify would have phase shift of more than $-180°$ and this would suggest stability. Indeed this is also the case, since it would give

$$\frac{G}{1+aG} = \frac{(s+1)(10^{-4}s+1)}{(s+1)(10^{-4}s+1)(10^{-1}s+1)+5\times 10^3}$$

which would have poles at -1.0005×10^4 and $(-0.0003 \pm 0.0224j) \times 10^4$. Since all poles have negative real parts, we'd expect this voltage regulator to be "stable". This still is probably not a very good regulator because the poles are lightly damped. In practice one would also add some additional phase lead into the feedback loop.